



Near ideals in near semigroups

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Abstract. In this paper, we introduced the notion of near subsemigroups, near ideals, near bi-ideals and homomorphisms of near semigroups on near approximation spaces. Then we give some properties of these near structures.

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1. Introduction

Rough sets were introduced by Z. Pawlak in his paper [16]. Algebraic structures of rough sets have been studied by many authors, for example, Bonikowaski [4], Iwinski [8], and Pomykala and Pomykala [24]. In 1994, Biswas and Nanda [3] introduced the notion of rough group and rough subgroups that their notion depends on the upper approximation and does not depend on the lower approximation. Miao et al. [14] improve definitions of rough group and rough subgroup, and prove their new properties. On the other hand, Kuroki and Wang [11] presented some properties of the lower and upper approximations with respect to the normal subgroups in 1996. In addition, some properties of the lower and the upper approximations with respect to the normal subgroups were studied in [5, 13, 26–28]. Also, Kuroki [12], introduced the notion of a rough ideal in a semigroup. Davvaz [6], introduced the notion of rough subring with respect to an ideal of a ring. Xiao and Zhang [30], studied the notions of rough prime ideals and rough fuzzy prime ideals in a semigroup. Bağirmaz and Özcan [1], studied the notion of rough semigroup on approximation space. Moreover, Bağirmaz [2], investigated rough prime ideals on approximation spaces.

Near sets were introduced by Peters [17] on the basis of a generalization of rough set theory. The algebraic properties of near sets are described in [19]. Recent work has considered near groups [9] and near semigroups [10]. The fundamental idea of near set theory is object description and classification according to perceptual knowledge. It is supposed that perceptual knowledge about objects is always given with respect to probe functions, i.e., real-valued functions which represent features of a physical object [7, 15, 20–23, 25, 29].

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The main purpose of this paper is to introduce near ideals and give some properties of such ideals on nearness approximation spaces. Finally, near image and near inverse image of near ideal are discussed. We introduced the notion of near ideal that our notion depends on the upper approximation and does not depend on the lower approximation. So, our definition of near ideal is similar to the definition of rough ideal [1].

2. Preliminaries

In this section, we will give some definitions and properties regarding near sets as in [18].

Table 1: Description Symbols

| Symbol | Interpretation |
|---------------|---|
| \mathbb{R} | Set of real numbers, |
| \mathcal{O} | Set of perceptual objects, |
| X | $X \subseteq \mathcal{O}$, set of sample objects, |
| x | $x \in \mathcal{O}$, sample objects, |
| F | a set of functions representing object features, |
| B | $B \subseteq F$, |
| Φ | $\Phi : \mathcal{O} \rightarrow \mathbb{R}^L$, object description, |
| L | L is a description length, |
| i | $i \leq L$, |
| $\Phi(x)$ | $\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \dots, \phi_i(x), \dots, \phi_L(x))$. |

Objects are known by their descriptions. An object description is defined by means of a tuple of function values $\Phi(x)$ associated with an object $x \in X$. The important thing to notice is the choice of functions $\phi_i \in B$ used to describe an object of interest. Assume that $B \subseteq F$ (see Table 1) is a given set of functions representing features of sample objects $X \subseteq \mathcal{O}$. Let $\phi_i \in B$, where $\phi_i : \mathcal{O} \rightarrow \mathbb{R}$. In combination, the functions representing object features provide a basis for an object description $\Phi_i : \mathcal{O} \rightarrow \mathbb{R}^L$, a vector containing measurements associated with each functional value $\phi_i(x)$, where the description length $|\Phi| = L$.

Object Description : $\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x), \dots, \phi_i(x), \dots, \phi_L(x))$.

The intuition underlying a description $\Phi(x)$ is a recording of measurements from sensors, where each sensor is modelled by a function ϕ_i .

Table 2: Nearness Approximation Space Symbols

| Symbol | Interpretation |
|----------------------------|--|
| B | $B \subseteq F$, |
| B_r | $r \leq B $ probe functions in B , |
| \sim_{B_r} | Indiscernibility relation defined using B_r , |
| $[x]_{B_r}$ | $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, equivalence class, |
| \mathcal{O} / \sim_{B_r} | $\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x' \in \mathcal{O}\}$, quotient set, |
| $\xi_{\mathcal{O}, B_r}$ | Partition $\xi_{\mathcal{O}, B_r} = \mathcal{O} / \sim_{B_r}$, |
| r | $\binom{ B }{r}$, i.e., $ B $ functions $\phi_i \in B$ taken r at a time, |
| $N_r(B)$ | $N_r(B) = \{\xi_{\mathcal{O}, B_r} \mid B_r \subseteq B\}$ set of partitions, |
| $N_r(B)_*X$ | $N_r(B)_*X = \bigcup_{x \in \mathcal{O}} \{[x]_{B_r} : [x]_{B_r} \subseteq X\}$, lower approximation, |
| $N_r(B)^*X$ | $N_r(B)^*X = \bigcup_{x \in \mathcal{O}} \{[x]_{B_r} : [x]_{B_r} \cap X \neq \emptyset\}$, upper approximation, |
| $Bnd_{N_r(B)}(X)$ | $N_r(B)^*X \setminus N_r(B)_*X = \{x \mid x \in N_r(B)^*X \text{ and } x \notin N_r(B)_*X\}$. |

A nearness approximation space (NAS) is denoted by $NAS = (\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ which is defined with a set of perceived objects \mathcal{O} , a set of probe functions F representing object features, an indiscernibility relation \sim_{B_r} defined relative to $B_r \subseteq B \subseteq F$, a collection of partitions (families of neighbourhoods) $N_r(B)$, and a neighbourhood overlap function N_r . The relation \sim_{B_r} is the usual indiscernibility relation from rough set theory restricted to a subset $B_r \subseteq B$. The subscript r denotes the cardinality of the restricted subset B_r , where we consider $\binom{|B|}{r}$, i.e., $|B|$ functions $i \in F$ taken r at a time to define the relation \sim_{B_r} . This relation defines a partition of \mathcal{O} into non-empty, pairwise disjoint subsets that are equivalence classes denoted by $[x]_{B_r}$, where $[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$. These classes form a new set called the quotient set \mathcal{O} / \sim_{B_r} , where $\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x' \in \mathcal{O}\}$. In effect, each choice of probe functions B_r defines a partition $\xi_{\mathcal{O}, B_r}$ on a set of objects \mathcal{O} , namely, $\xi_{\mathcal{O}, B_r} = \mathcal{O} / \sim_{B_r}$. Every choice of the set B_r leads to a new partition of \mathcal{O} . Let F denote a set of features for objects in a set X , where each $\phi_i \in F$ that maps X to some value set V_{ϕ_i} (range of ϕ_i). The value of $\phi_i(x)$ is a measurement associated with a feature of an object $x \in X$. The overlap function v_{N_r} is defined by $v_{N_r} : P(\mathcal{O}) \times P(\mathcal{O}) \rightarrow [0, 1]$, where $P(\mathcal{O})$ is the powerset of \mathcal{O} . The overlap function v_{N_r} maps a pair of sets to a number in $[0, 1]$ representing the degree of overlap between sets of objects with features defined by probe functions $B_r \subseteq B$. For each subset $B_r \subseteq B$ of probe functions, define the binary relation $\sim_{B_r} = \{(x, x') \in \mathcal{O} \times \mathcal{O} : \forall \phi_i \in B_r, \phi_i(x) = \phi_i(x')\}$. Since each \sim_{B_r} is, in fact, the usual indiscernibility relation [16], for $B_r \subseteq B$ and $x \in \mathcal{O}$, let $[x]_{B_r}$ denote the equivalence class containing x , i.e., $[x]_{B_r} = \{x' \in \mathcal{O} \mid \forall \phi \in B_r, \phi(x') = \phi(x)\}$. If $(x, x') \in \sim_{B_r}$ (also written $x \sim_{B_r} x'$), then x and x' are said to be B indiscernible with respect to all feature probe functions in B_r . Then define a collection of partitions $N_r(B)$ (families of neighborhoods), where $N_r(B) = \{\xi_{\mathcal{O}, B_r} \mid B_r \subseteq B\}$. Families of neighborhoods are constructed for each combination of probe functions in B using $\binom{|B|}{r}$, i.e., $|B|$ probe functions taken r at a time.

Proposition 1. [10] Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and $X, Y \subseteq \mathcal{O}$. Then, the approximations have the following properties:

- (1) $N_r(B)_*X \subseteq X \subseteq N_r(B)^*X$,
- (2) $N_r(B)^*(X \cup Y) = N_r(B)^*X \cup N_r(B)^*Y$,
- (3) $N_r(B)_*(X \cap Y) = N_r(B)_*X \cap N_r(B)_*Y$,
- (4) $X \subseteq Y$ implies $N_r(B)_*X \subseteq N_r(B)_*Y$,
- (5) $X \subseteq Y$ implies $N_r(B)^*X \subseteq N_r(B)^*Y$,
- (6) $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*X \cap N_r(B)^*Y$,
- (7) $N_r(B)_*(X \cup Y) \supseteq N_r(B)_*X \cup N_r(B)_*Y$.

A nonempty subset H of a semigroup S is said to be a subsemigroup of S , if $ab \in H$ for all $a, b \in H$, i.e., $H^2 \subseteq H$. A nonempty subset I of a semigroup S is said to be a left (resp. right) ideal of S if $SI \subseteq I$ (resp. $IS \subseteq I$). A nonempty subset I of S is called an ideal of S if I is both a left and a right ideal of S . A subsemigroup H of S is called a bi-ideal of S if $HSH \subseteq H$.

Let S be a semigroup. An element $x \in S$ is a left identity of S , if $\forall y \in S : xy = y$. Similarly, x is a right identity of S , if $\forall y \in S : yx = y$. If x is both a left and a right identity of S , then x is called an identity of S . A semigroup is a monoid, if it has an identity. The identity of a monoid S is usually denoted by 1_S , or just by 1 , for short. A monoid G is a group, if every $x \in G$ has a (group) inverse $x^{-1} \in G : xx^{-1} = 1 = x^{-1}x$.

3. Near ideals

In this section, we introduce the notions of near subsemigroup, near ideal and near bi-ideal on a near approximation space, and study some of its properties.

Definition 1. [10] Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and let (\cdot) be a binary operation defined on \mathcal{O} .

A subset S of the set of perceptual objects \mathcal{O} is called a near semigroup on nearness approximation space, provided the following properties are satisfied:

- (1) For all $x, y \in S$, $x \cdot y \in N_r(B)^*S$,
- (2) For all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*S$.

Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let S be a near semigroup. There is only one an element $x \in N_r(B)^*S$ is a left identity of S , if $\forall y \in S : xy = y$. Similarly, x is a right identity of S , if $\forall y \in S : yx = y$. If x is both a left and a right identity of S , then x is called a near identity of S . A near semigroup is a near monoid, if it has a near identity.

Lemma 1. *A near semigroup S can have at most one identity. In fact, if S has a left identity x and a right identity y , then $x = y$. In particular, the identity of a near monoid is unique.*

Proof. By the definitions, $y = xy = x$. ■

The identity of a near monoid S is denoted by e . A near monoid G is a near group, if every $x \in G$ has a inverse $x^{-1} \in G$:

$$x^{-1}x = e = x^{-1}x.$$

Definition 2. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let S be a near semigroup and H a nonempty subset of S . A nonempty subset H of a near semigroup S is said to be a near subsemigroup of S , if $ab \in N_r(B)^*H$ for all $a, b \in H$, i.e., $HH \subseteq N_r(B)^*H$.

Another difference between near semigroup and semigroup is the following:

Proposition 2. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let H_1 and H_2 be two near subsemigroups of the near semigroup S . A sufficient condition for intersection of two near subsemigroups of a near semigroup to be a near subsemigroup is $N_r(B)^*H_1 \cap N_r(B)^*H_2 = N_r(B)^*(H_1 \cap H_2)$.

Proof. Suppose H_1 and H_2 are two near subsemigroups of S . It is obvious that $H_1 \cap H_2 \subseteq S$. Consider $x, y \in H_1 \cap H_2$. Because H_1 and H_2 are near subsemigroups, we have $xy \in N_r(B)^*H_1$, $xy \in N_r(B)^*H_2$, i.e. $xy \in N_r(B)^*H_1 \cap N_r(B)^*H_2$. Assuming $N_r(B)^*H_1 \cap N_r(B)^*H_2 = N_r(B)^*(H_1 \cap H_2)$, we have $xy \in N_r(B)^*(H_1 \cap H_2)$. Thus $H_1 \cap H_2$ is a near subsemigroup of S . ■

Definition 3. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . A nonempty subset I of a near semigroup S is said to be a near left (resp. right) ideal of S if $SI \subseteq N_r(B)^*I$ (resp. $IS \subseteq N_r(B)^*I$). A nonempty subset I of S is called an near ideal of S if I is both a left and a right near ideal of S .

Proposition 3. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} and $S \subseteq \mathcal{O}$. Then

- (1) If H is a subsemigroup of semigroup S , then H is a near subsemigroup of near semigroup S .
- (2) If I is a left (right, two-sided) ideal of semigroup S , then I is a near left (right, two-sided) ideal of near semigroup S .

Proof. (1) Let H be a subsemigroup of semigroup S , that is, $HH \subseteq H$. By Proposition 1 (1), we have that $H \subseteq N_r(B)^*H$. Thus $HH \subseteq N_r(B)^*H$. Hence, H is a near subsemigroup of near semigroup S .

(2) Let I be a left ideal of semigroup S , that is, $SI \subseteq I$. Since $I \subseteq S$, by Proposition 1 (5), we know that $N_r(B)^*I \subseteq N_r(B)^*S$. Then, by Proposition 1.(1), we have that

$I \subseteq N_r(B)^*I$. Thus $SI \subseteq I \subseteq N_r(B)^*I$. This means that I is a near left ideal of near semigroup S . Also, we can easily show that I is a near right ideal of near semigroup S . The other cases can be seen in a similar way.

■

The following example shows that the converse of by Proposition 3 is not true.

Example 1. Let $\mathcal{O} = \{1, 2, 3, 4, 5\}$ be a set of perceptual objects with the following multiplication table 2 and $B = \{\phi_1, \phi_2, \phi_3\} \subseteq F$ be a set of probe functions with the following multiplication table 3, respectively.

| \cdot | a | b | c | d | e | f |
|---------|-----|-----|-----|-----|-----|-----|
| a | a | a | a | b | b | a |
| b | a | b | b | c | d | f |
| c | c | c | c | e | d | f |
| d | b | a | b | d | f | a |
| e | a | b | c | d | e | f |
| f | b | f | b | f | f | a |

Table 2.

| | a | b | c | d | e | f |
|----------|-----|-----|-----|-----|-----|-----|
| ϕ_1 | 1 | 1 | 2 | 3 | 2 | 4 |
| ϕ_2 | 1 | 2 | 2 | 3 | 4 | 4 |
| ϕ_3 | 1 | 1 | 3 | 2 | 4 | 4 |

Table 3.

Since

$$[a]_{\phi_1} = \{a, b\}, [c]_{\phi_1} = \{c, e\}, [d]_{\phi_1} = \{d\}, [f]_{\phi_1} = \{f\}$$

we have

$$\xi_{\phi_1} = \{[a]_{\phi_1}, [c]_{\phi_1}, [d]_{\phi_1}, [f]_{\phi_1}\}.$$

Since

$$[a]_{\phi_2} = \{a\}, [b]_{\phi_2} = \{b, c\}, [d]_{\phi_2} = \{d\}, [e]_{\phi_2} = \{e, f\}$$

we have

$$\xi_{\phi_2} = \{[a]_{\phi_2}, [b]_{\phi_2}, [d]_{\phi_2}, [e]_{\phi_2}\}.$$

Since

$$[a]_{\phi_3} = \{a, b\}, [c]_{\phi_3} = \{c\}, [d]_{\phi_3} = \{d\}, [e]_{\phi_3} = \{e, f\}.$$

we have

$$\xi_{\phi_3} = \{[a]_{\phi_3}, [c]_{\phi_3}, [d]_{\phi_3}, [e]_{\phi_3}\}.$$

Therefore, for $r = 1$, a set partitions of \mathcal{O} is $N_1(B) = \{\xi_{\phi_1}, \xi_{\phi_2}, \xi_{\phi_3}\}$. Let $S = \{b, c, d\}$ be a subset of perceptual \mathcal{O} as in table 4.

| \cdot | b | c | d |
|---------|-----|-----|-----|
| b | b | b | c |
| c | c | c | e |
| d | a | b | d |

Table 4.

Then, we have that

$$\begin{aligned} N_1(B)^*S &= \bigcup_{x \in \mathcal{O}} \{[x]_{B_1} : [x]_{B_1} \cap S \neq \emptyset\} \\ &= \{a, b\} \cup \{c, e\} \cup \{d\} \cup \{b, c\} \cup \{d\} \cup \{c\} \\ &= \{a, b, c, d, e\}. \end{aligned}$$

From Definition 1, $S \subseteq \mathcal{O}$ is a near semigroup. Because $d \cdot b = a \notin S$ we have S is not a semigroup.

Let $H = \{b, c\}$, then

$$\begin{aligned} N_1(B)^*H &= \bigcup_{x \in \mathcal{O}} \{[x]_{B_1} : [x]_{B_1} \cap H \neq \emptyset\} \\ &= \{a, b\} \cup \{c, e\} \cup \{a, b, c\} \\ &= \{a, b, c\}. \end{aligned}$$

From Definition 2, H is a near subsemigroup of near semigroup S .

Let $I = \{c, d\}$, then

$$\begin{aligned} N_1(B)^*I &= \bigcup_{x \in \mathcal{O}} \{[x]_{B_1} : [x]_{B_1} \cap I \neq \emptyset\} \\ &= \{c, e\} \cup \{d\} \cup \{b, c\} \cup \{c\} \\ &= \{b, c, d, e\}. \end{aligned}$$

From Definition 3, I is a near left ideal of near semigroup S .

The Proposition 3 shows that the notion of a near semigroup (left ideal, right ideal, two-sided ideal) is an extended notion of an ordinary semigroup (left ideal, right ideal, two-sided ideal).

Another difference between near left (right, two-sided) ideal and left (right, two-sided) ideal is the following:

Proposition 4. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let I_1 and I_2 be two left (right, two-sided) ideals of the near semigroup S . A sufficient condition for intersection of two left (right, two-sided) ideal of a near semigroup to be a near left (right, two-sided) ideal is $N_r(B)^*I_1 \cap N_r(B)^*I_2 = N_r(B)^*(I_1 \cap I_2)$.

Proof. Suppose I_1 and I_2 are two near left ideals of S . It is obvious that $I_1 \cap I_2 \subset S$. Consider $x \in S$ and $y \in I_1 \cap I_2$. Because I_1 and I_2 are near left ideals, we have $xy \in N_r(B)^*I_1$, $xy \in N_r(B)^*I_2$, i.e. $xy \in N_r(B)^*I_1 \cap N_r(B)^*I_2$. Assuming $N_r(B)^*I_1 \cap N_r(B)^*I_2 = N_r(B)^*(I_1 \cap I_2)$, we have $xy \in N_r(B)^*(I_1 \cap I_2)$. Thus $I_1 \cap I_2$ is a near left ideal of S . The other cases can be seen in a similar way. ■

Definition 4. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . A near subsemigroup I of near semigroup S is called a near bi-ideal of S if $ISI \subseteq N_r(B)^*I$.

Proposition 5. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let $S \subseteq \mathcal{O}$ be a semigroup. If I is a bi-ideal of S , then I is a near bi-ideal of near semigroup S .

Proof. Let I be a bi-ideal of S , i.e., $ISI \subseteq I$. Since $I \subseteq S$, by Proposition 1 (5), we know that $N_r(B)^*I \subseteq N_r(B)^*S$. Then, by Proposition 1 (1), we have that $I \subseteq N_r(B)^*I$. Hence, I is a near bi-ideal of near semigroup S . ■

Lemma 2. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} . Let $X \subseteq \mathcal{O}$. Then $N_r(B)^*(N_r(B)^*X) = N_r(B)^*(X)$.

Proof. Since $N_r(B)^*(X) \subseteq N_r(B)^*(X)$, then, by Proposition 1 (1) $N_r(B)^*(X) \subseteq N_r(B)^*(N_r(B)^*X)$. Conversely, let $[x]_{B_r} \in N_r(B)^*(N_r(B)^*X)$. Then $[x]_{B_r} \cap N_r(B)^*X \neq \emptyset$. Thus $[x]_{B_r} \in N_r(B)^*X$. Hence $N_r(B)^*(N_r(B)^*X) = N_r(B)^*(X)$. ■

Proposition 6. Let $(\mathcal{O}, F, \sim_{B_r}, N_r, v_{N_r})$ be a nearness approximation space and (\cdot) be a binary operation defined on \mathcal{O} and $S \subseteq U$ be a near semigroup. If I is a near right ideal of S and J is a near left ideal of S , then

$$N_r(B)^*(IJ) \subseteq N_r(B)^*I \cap N_r(B)^*J.$$

Proof. Let I be a near right ideal of S and J be a near left ideal of S , then $IJ \subseteq IS \subseteq N_r(B)^*I$ and $IJ \subseteq SJ \subseteq N_r(B)^*J$. Thus $IJ \subseteq N_r(B)^*I \cap N_r(B)^*J$. Thus, it follows from Proposition 1.(5) and (6) that

$$\begin{aligned} N_r(B)^*(IJ) &\subseteq N_r(B)^*(N_r(B)^*I \cap N_r(B)^*J) \\ &\subseteq N_r(B)^*(N_r(B)^*I) \cap N_r(B)^*(N_r(B)^*J). \end{aligned}$$

Then, by Lemma 2, we have that $N_r(B)^*(N_r(B)^*I) = N_r(B)^*(I)$ and $N_r(B)^*(N_r(B)^*J) = N_r(B)^*(J)$. Hence $N_r(B)^*(IJ) \subseteq N_r(B)^*(I) \cap N_r(B)^*(J)$. ■

4. Homomorphisms of near semigroups

Let $(\mathcal{O}_1, F_1, \sim_{B_{r_1}}, N_{r_1}, v_{N_{r_1}})$, $(\mathcal{O}_2, F_2, \sim_{B_{r_2}}, N_{r_2}, v_{N_{r_2}})$ be two nearness approximation spaces, and $(\cdot), (\circ)$ be binary operations over universes \mathcal{O}_1 and \mathcal{O}_2 , respectively.

Definition 5. Let $S_1 \subset \mathcal{O}_1$, $S_2 \subset \mathcal{O}_2$ be near semigroups. If there exists a surjection $\phi : N_{r_1}(B)^*(S_1) \rightarrow N_{r_2}(B)^*(S_2)$ such that $\phi(x \cdot y) = \phi(x) \circ \phi(y)$ for all $x, y \in N_{r_1}(B)^*(S_1)$ then ϕ is called a near homomorphism and S_1, S_2 are called near homomorphic semigroups.

Proposition 7. Let S_1 and S_2 be near homomorphic semigroups. If (\cdot) satisfies the commutative law, then (\circ) also satisfies it.

Proof. Consider S_1, S_2 , and ϕ such that $\phi(x \cdot y) = \phi(x) \circ \phi(y)$ for all $x, y \in G_1$. For every $\phi(x), \phi(y) \in S_2$, since ϕ is surjection, there exist $x, y \in S_1$ such that $x \mapsto \phi(x), y \mapsto \phi(y)$. Thus $\phi(x \cdot y) = \phi(x) \circ \phi(y)$, and $\phi(y \cdot x) = \phi(y) \circ \phi(x)$. Now, assuming $x \cdot y = y \cdot x$, we obtain $\phi(x) \circ \phi(y) = \phi(y) \circ \phi(x)$. That means that (\circ) satisfies the commutative law.

■

Proposition 8. Let $S_1 \subset \mathcal{O}_1, S_2 \subset \mathcal{O}_2$ be near homomorphic semigroups and let $N_{r_2}(B)^*(\phi(S_1)) = N_{r_2}(B)^*(S_2)$. Then $\phi(S_1)$ is a near semigroup.

Proof. (1) $\forall x', y' \in \phi(S_1)$, consider $x, y \in S_1$ such that $x \mapsto x', y \mapsto y'$. We have $\phi(x \cdot y) = \phi(x) \circ \phi(y) \in N_{r_2}(B)^*(S_2) = N_{r_2}(B)^*(\phi(S_1))$, that is $x' \circ y' \in N_{r_2}(B)^*(\phi(S_1))$.

(2) S_1 is a near semigroup, so $\forall x, y, z \in S_1, x \cdot (y \cdot z) = (x \cdot y) \cdot z$. Hence,

$$\phi(x \cdot (y \cdot z)) = \phi(x) \circ \phi(y \cdot z) = \phi(x) \circ (\phi(y) \circ \phi(z))$$

$$\phi((x \cdot y) \cdot z) = \phi(x \cdot y) \circ \phi(z) = (\phi(x) \circ \phi(y)) \circ \phi(z),$$

$$\text{i.e., } (\phi(x) \circ \phi(y)) \circ \phi(z) = \phi(x) \circ (\phi(y) \circ \phi(z)).$$

Consequently, we can conclude that $\phi(S_1)$ is a near semigroup. ■

Proposition 9. Let $S_1 \subset \mathcal{O}_1, S_2 \subset \mathcal{O}_2$ be near homomorphic semigroups and let H, I be a near subsemigroup and a near left (right, two-sided) ideal of S_1 , respectively. Then;

(a) $\phi(H)$ is a near subsemigroup of S_2 if $\phi(N_{r_1}(B)^*(H)) = N_{r_2}(B)^*(\phi(H))$,

(b) $\phi(I)$ is a near left (right, two-sided) ideal of S_2 if $\phi(N_{r_1}(B)^*(I)) = N_{r_2}(B)^*(\phi(I))$.

Proof. (a) Consider a onto mapping ϕ from $N_{r_1}(B)^*(S_1)$ to $N_{r_2}(B)^*(S_2)$ such that $\forall x, y \in N_{r_1}(B)^*S_1, \phi(x \cdot y) = \phi(x) \circ \phi(y)$. For all $\phi(x), \phi(y) \in \phi(H)$, by the definition of ϕ , there exists $x, y \in H$ such that $x \mapsto \phi(x)$ and $\phi(x) \circ \phi(y) = \phi(x \cdot y) \in \phi(N_{r_1}(B)^*H)$. Since $\phi(N_{r_1}(B)^*H) = N_{r_2}(B)^*\phi(H)$, we have $\phi(x) \circ \phi(y) \in N_{r_2}(B)^*\phi(H)$.

(b) By (a), it is easy to see that $\phi(I)$ is a near subsemigroup of S_2 if $\phi(N_{r_1}(B)^*I) = N_{r_2}(B)^*\phi(I)$. Since $\forall \phi(x) \in S_2, \phi(y) \in \phi(I)$ there is $\phi(x) \circ \phi(y) = \phi(x \cdot y)$ and I is a near left ideal of S_1 , we have $x \cdot y \in N_{r_1}(B)^*I$. Thus $\phi(x \cdot y) \in \phi(N_{r_1}(B)^*I)$. Since $\phi(N_{r_1}(B)^*(I)) = N_{r_2}(B)^*(\phi(I))$, we have $\phi(x) \circ \phi(y) \in N_{r_2}(B)^*\phi(I)$. Hence $\phi(I)$ is a rough left ideal of S_2 .

Similarly, we can prove the other statement. ■

Proposition 10. Let $S_1 \subset \mathcal{O}_1, S_2 \subset \mathcal{O}_2$ be near homomorphic semigroups and let I be a near bi-ideal of S_1 . Then, $\phi(I)$ is a near bi-ideal of S_2 if $\phi(N_{r_1}(B)^*I) = N_{r_2}(B)^*\phi(I)$.

Proof. By Proposition 9 item (a), $\phi(I)$ is a near subsemigroup of S_2 if $\phi(N_{r_1}(B)^*(I)) = N_{r_2}(B)^*(\phi(I))$. Since $\forall \phi(x) \in S_2, \phi(y) \in \phi(I)$ there is $\phi(y) \circ \phi(x) \circ \phi(y) = \phi(y \cdot x \cdot y)$ and I is a near bi-ideal of S_1 , we have $y \cdot x \cdot y \in N_{r_1}(B)^*(I)$. Thus $\phi(y \cdot x \cdot y) \in \phi(N_{r_1}(B)^*I)$. Since $\phi(N_{r_1}(B)^*I) = N_{r_2}(B)^*\phi(I)$, we have $\phi(y) \circ \phi(x) \circ \phi(y) \in N_{r_2}(B)^*\phi(I)$. Hence $\phi(I)$ is a near bi-ideal of S_2 . ■

Proposition 11. Let $S_1 \subset \mathcal{O}_1, S_2 \subset \mathcal{O}_2$ be near homomorphic semigroups and let H_2, I_2 be a near subsemigroup and a near left (right, two-sided) ideal of S_2 . Then,

(a) $\phi^{-1}(H_2) = H_1$ is a near subsemigroup of S_1 if $\phi(N_{r_1}(B)^*(H_1)) = N_{r_2}(B)^*(\phi(H_1))$,

(b) $\phi^{-1}(I_2) = I_1$ is a near left (right, two-sided) ideal of S_1 if $\phi(N_{r_1}(B)^*(I_1)) = N_{r_2}(B)^*(\phi(I_1))$.

Proof. (a) Since $\phi^{-1}(H_2) = H_1$, we have $\phi(H_1) = H_2$, and so $N_{r_2}(B)^*H_2 = N_{r_2}(B)^*(\phi(H_1)) = \phi(N_{r_1}(B)^*H_1)$. $\forall x, y \in H_1$, we have $\phi(x), \phi(y) \in H_2$. Since H_2 is a near subsemigroup, we get $\phi(x) \circ \phi(y) \in N_{r_2}(B)^*H_2$. Thus $\phi(x \cdot y) \in \phi(N_{r_1}(B)^*H_1)$. Thus $x \cdot y \in N_{r_1}(B)^*H_1$.

(b) By (a), it is easy to see that $\varphi^{-1}(I_2) = I_1$ is a near subsemigroup of S_1 if $\phi(N_{r_1}(B)^*I_1) = N_{r_2}(B)^*\phi(I_1)$. Since $\phi^{-1}(I_2) = I_1$, we have $\phi(I_1) = I_2$, and so $N_{r_2}(B)^*I_2 = N_{r_2}(B)^*\phi(I_1) = \phi(N_{r_1}(B)^*I_1)$. $\forall x \in S_1$ and $y \in I_1$, we have $\phi(x) \in \phi(S_1)$ and $\phi(y) \in I_2$. Since I_2 is a near left ideal of S_2 , we have $\phi(x) \circ \phi(y) \in N_{r_2}(B)^*I_2$. Thus $\phi(x \cdot y) \in \phi(N_{r_1}(B)^*I_1)$. Thus $x \cdot y \in N_{r_1}(B)^*I_1$.

Therefore, $\varphi^{-1}(I_2) = I_1$ is a near left ideal of S_1 . Similarly, we can prove the other statement. ■

Proposition 12. Let $S_1 \subset \mathcal{O}_1$, $S_2 \subset \mathcal{O}_2$ be near homomorphic semigroups and let I_2 be a near bi-ideal of S_2 . Then, $\phi^{-1}(I_2) = I_1$ is a near bi-ideal of S_1 if $\phi(N_{r_1}(B)^*I_1) = N_{r_2}(B)^*\phi(I_1)$.

Proof. By Proposition 11 item (a), $\phi^{-1}(I_2) = I_1$ is a near subsemigroup of S_1 if $\phi(N_{r_1}(B)^*I_1) = N_{r_2}(B)^*\phi(I_1)$. Since $\varphi^{-1}(I_2) = I_1$, we have $\varphi(I_1) = I_2$, and so $N_{r_2}(B)^*I_2 = N_{r_2}(B)^*\phi(I_1) = \phi(N_{r_1}(B)^*I_1)$. $\forall x \in S_1$ and $y \in I_1$, we have $\phi(x) \in \phi(S_1)$ and $\phi(y) \in I_2$. Since I_2 is a near bi-ideal of S_2 , we have $\phi(y) \circ \phi(x) \circ \phi(y) \in N_{r_2}(B)^*I_2$. Thus $\phi(y \cdot x \cdot y) \in \phi(N_{r_1}(B)^*I_1)$. Thus $y \cdot x \cdot y \in N_{r_1}(B)^*I_1$.

Therefore, $\phi^{-1}(I_2) = I_1$ is a near bi-ideal of S_1 . ■

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